FUNCTIONS OF EXPLANATIONS AND DIMENSIONS OF RATIONALITY: COMBINING FRAMEWORKS

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This paper presents first results from a research project aimed at combining two theoretical frameworks, one concerning explanations and one related to rationality. The two theoretical lenses are used to understand one episode from a teaching experiment carried out in grade seven, concerning the construction of rectangles with a given perimeter. The combination of frameworks allows a finer analysis of the teaching episode and extends the original frameworks.

INTRODUCTION

Various theoretical frameworks are available to mathematics education researchers interested in analysing complex mathematical activities such as conjecturing, proving, and modelling. Sometimes, the networking of different theories may improve the understanding of data (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). This paper combines two frameworks, one related to functions of explanations (Levenson, Barkai, & Larson, 2013) and one related to rational behavior in conjecturing and proving (Boero & Morselli, 2009), in order to investigate classroom tasks and didactical sequences which promote conjecturing and proving among students.

We chose to combine the two frameworks for several reasons. First, the framework related to the functions of explanation was previously used to analyse tasks found in national guidelines and curricula. We were interested in investigating the use of this framework when analysing classroom tasks and didactical sequences. The model of rationality was initially developed for the analysis of students' processes when faced with conjecturing and proving tasks. Those tasks usually took the form of "What can you tell about…?". We were interested in investigating the use of this model when students are faced with other kind of tasks, such as inquiry-based tasks, or tasks which require them to explain procedures. Finally, we wished to examine the possible links between functions of explanations and dimensions of rationality.

THEORETICAL FRAMEWORK S

Functions of explanations

Explanations are used every day in the mathematics classroom and are an integral part of learning and teaching mathematics. However, many research studies use the term 'explanation' in different ways, alluding to different functions of explanations (e.g., Hemmi, Lepik, & Viholainen, 2013; Yackel, 2001). Levenson and Barkai (2013), and Levenson, Barkai, and Larson (2013) set out to systemize and classify the possible functions of explanations which may arise from solving mathematical tasks in the

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classroom. Analysing curriculum documents in Israel and in Sweden led to six possible functions:

Function 1: Explanation as a description of one's thinking process or way of solving a problem (i.e., How did you solve the problem? Explain.)

Function 2: Explanation as an answer to a "why" question where the underlying assumption is that the explanation should rely on mathematical properties and generalizations (i.e., Why is this statement true/false? Explain.)

Function 3: Explanations as interpretations (i.e., Explain what this mathematical statement means in an everyday context. Explain an everyday occurrence in a mathematical context.)

Function 4: Explanations as a step in directing new explorations leading to generalizations (i.e., Find all possible solutions and explain.)

Function 5: Explanation as justifying the reasonableness or plausibility of a strategy or solution (i.e., Why did I choose to solve the problem in this way?)

Function 6: Explanations as a means of communication. This function may be a more general function considering that explanations, whether written down or expressed orally, are meant to be communicated.

It should be noted that the function of an explanation may depend on the task given as well as the context in which an explanation is requested or given. In the Israeli curriculum, for example, it was found that an investigative task may call for a child to explain a solution with the possible aim that this explanation leads to further investigation. A different task may call for an explanation which merely describes how to solve the task. It might be that the same task, implemented in different ways by the teacher, could lead to different functions of explanations. In Sweden, the functions of explanations seem to be tied in with major aims for teaching mathematics in primary school. The current study extends the study by Levenson, Barkai, and Larson (2013) by attempting to use their classification of the functions of explanations when analysing a series of classroom tasks given in an Italian mathematics classroom and by combining it with the framework of rationalization set out by Boero & Morselli (2009).

Rationality

Boero & Morselli (2009) developed a theoretical model for proving as a rational behaviour, derived from the construct of rationality proposed by Habermas (2003). According to the model, the discursive practice of proving may be seen as made up of three interrelated components: an epistemic rationality (ER), related to the conscious validation of statements according to shared premises and legitimate ways of reasoning; a teleological rationality (TR), inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity; and a communicative rationality (CR), inherent in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture. The construct was further developed to analyse specific phases within the conjecturing process, for instance the use of algebraic language (Morselli & Boero, 2011).

The following episode, taken from a proving task (Morselli & Boero, 2011) illustrates the three components of rational behaviour, highlighting teleological rationality. Seventh grade students were given the following problem: "The teacher proposes a game: Choose a number, double it, add 5, take away the chosen number, add 8, take away 2, take away the chosen number. Without knowing the result of the game, is it possible for the teacher to guess the number that you initially chose? If yes, in what way?". Students played the game and gradually discovered that the teacher always guesses ten. When explaining why the result is always ten, two solutions emerged: the expression ($N\times2+5-N+8-2-N-1=10$) and the sequence of calculations ($N\times2 = A$; $A+5= B$; $B-N=C$; $C+8=D$; $D-2=E$; $E-N=F$; $F-1=10$). Comparing the two representations sheds light on the dimensions on rationality. The two representations are correct from a mathematical point of view, thus fulfill the requirement of epistemic rationality, and are clear from the communicative point of view. Both representations lead to the result of the game, but the first one is more efficient in terms of the goal of the activity (showing that the result is always 10), since it shows that N is at first doubled and then taken away twice, making no contribution to the final result. From a teleological point of view, the first representation is more appropriate. The teleological component of rationality refers to efficiency and usefulness in relation to the final goal one wishes to achieve (here, showing that N does not affect the final result). We refer to teleological rationality for all those strategic choices that are linked to the final aim of the activity.

Combining the frameworks

As previously mentioned, the construct of rationality was initially adapted from a general description referring to any discursive practice (not only within mathematics), to the process of proving. The process of proving relates to only one specific function of explanation – explaining why a statement holds true. Yet explanations may have several functions. The aim of this study is to explore the possibility of extending the scope of the rationality construct and describe the rationality at issue when explaining. Can we see an expression of the three dimensions of rationality in explanation processes? If so, how can each dimension be described in relation to each function of explanations?

METHOD

The first step in combining two frameworks is to understand each one separately. Mutual understanding was achieved by reading previous research reports and by a first cycle of data analysis. Each author analysed the data by means of the two theoretical lenses. The two analyses were compared and divergent interpretations were questioned, so as to promote mutual understanding (of the frameworks) and a more complete interpretation of the teaching episode. Finally, we worked together in developing a combined description of rationality in explanation processes. In the following sections we describe some background of the project where this study was set and analyse some written productions in terms of the functions of explanations which arose and the dimensions of rationality observed.

TASK SEQUENCE: ISOPERIMETRIC RECTANGLES

The episode we refer to comes from a teaching experiment carried out in 2012 in a lower secondary school, within the project "Language and Argumentation", aimed at designing and experimenting task sequences with a special focus on argumentation and proof (Morselli, 2013). Core tasks are usually proposed as open-ended questions (*What can you tell about…?*) where, according to the socio-mathematical norms of the class, each answer must be justified.

The data below was collected from the beginning parts of the task sequence "*Isoperimetric rectangles*", implemented in grade 7 (age of the students: 13-14). At the core of this activity is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the square has the maximum area. The task sequence began with an explorative task in paper and pencil: "*Draw four rectangles with a perimeter of 20 cm*". After drawing the rectangles, students were asked to reflect on the construction of their rectangles. The second task, to be worked on in groups, was: *Compare the methods you used to draw the rectangles and synthesize*. Here we present our analysis of the group work.

FINDINGS

Students' written collective responses were collected by the researcher. Below, we analyse the results of four groups, using the dual lens of functions of explanations and rationality. For clarity of analysis, we break up the students' writing into segments (Seg a, Seg b).

Group One

Two students working together wrote the following:

Seg a: In order to make rectangles with a perimeter of 20 cm one must make 10 cm and then multiply by 2.

Seg b: With this method one can make 9 rectangles: 6+4, 7+3, 8+2, 9+1, 4+6, 3+7, 2+8 and 1+9, but the first, second, third and fourth one are equal to the last four. 5+5 cannot be done because a square is made.

Segment a: Regarding functions of explanations, the students describe their method. This is Function 1. It might be said that the first segment, in which the students explained their method, led them to explore several options, basically covering all the whole number options (Seg b). Evidence of this may be seen in what the students wrote in Seg b, "With this method one can make…" In addition, the students set out what is, in their opinion, all the possible rectangles taking into consideration the constrictions. In this sense, we claim that Seg a may be related to Function 4 in that it led to additional exploration. In addition, the students justify their strategy. They explain that they are looking for numbers which sum to 10 cm in order to find a perimeter of 20 cm. Therefore, Function 5 is present. Regarding modes of rationality, the students began their explanation by writing "In order to….". This clear indication of working towards a goal is evidence of Teleological Rationality (TR). The explanation is correct

(Epistemic Rationality – ER) and communicated in a comprehensible although not complete manner (Communicative Rationality – CR). For example, although the students noted that “one must make 10 cm”, this statement is rather general; it was not explicitly stated that it is necessary to take exactly two numbers whose sum is ten.

Segment b: Regarding functions of explanations, there might be evidence of Function 2 in that the students explain why they do not include 5+5. It is likely that students compare the properties of squares and rectangles and incorrectly conclude that a rectangle must have unequal sides. This incorrect property is used for justifying that the solution 5+5 is not acceptable. Regarding rationality, the students only list rectangles with whole number lengths and they do not include the square. They explicitly state that there are nine rectangles which fit the requirements of the problem, implying that these are all the possible rectangles. Because the square is not included, there is a lack of ER. Furthermore, their explanation consists mostly of examples without further elaboration (CR). However, there is a clear goal to list all possible rectangles and the students organize their discourse accordingly (TR).

Group Two

Two students working together wrote the following:

Seg a: We looked for a number that gave 10 and then we added it, for example 8+2. Afterwards we added the same number 8+8 is the length and 2+2 is the side thus giving a rectangle.

Seg b: Ex 9+9 and 1+1 = 20 and 3+3 [under the numbers it is written “sides”] and 7+7 [under the numbers it is written “bases”] =20 ex3 6+6 [under the numbers: “bases”] and 4+4 [under the numbers: “sides”] = 20.

Segment *a*: Regarding functions of explanations, the students describe what they are doing (Function 1). Unlike with Group one, it does not seem that the students expand their exploration beyond giving a few more examples (Seg *b*). Thus, there is no evidence of Function 4. Nor do they justify their strategy of looking for numbers that add to 10 cm. Therefore, Function 5 is not present. As in Group one, the students have a clear goal and they state so explicitly, “We looked for…” (TR). Their procedure is correct (ER), however, like Group one, their communication lacks the necessary details (CR). Instead of writing that they looked for two numbers which add to 10, they wrote, “We looked for a number that gave 10.” In addition, the word “gave” is not mathematical and does not convey that the students are looking for numbers which “add” to ten.

Segment *b*: Regarding functions of explanations, it seems that only Function 1 was present. Regarding rationality, the examples are correct and because the students do not claim that they have found all possible rectangles with perimeter 20 cm, we cannot comment on and evaluate the lack of additional examples (ER). On the one hand, the students attempt to communicate their ideas in a clear manner by designating “sides” and “bases” (CR) although perhaps, mathematically, it would be more precise to distinguish between “heights” and “bases” (ER). On the other hand, the statement “9+9

and 1+1=20" does not conform to mathematical standards of communication (e.g., using "and" instead of "+") (CR).

Group Three

Three students worked together and wrote the following:

Seg a: We added two different sides, whose sum was 10 which multiplied by two the result was 20, that is to say the perimeter of the rectangle.

Seg b: Other rectangles with perimeter of 20 cm, are not possible, unless with these sides 6cm+4cm= x2, 8+2cm= x2, 9+1cm= x2, 7+3cm= x2.

5+5+5+5cm=20 cm, it is not a rectangle, but a square. 10+10 cm= 20 cm but it is not a rectangle.

Segment *a*: Regarding functions of explanation, in addition to describing what they did (Function 1), the students justified why their strategy is valid – because it leads to a rectangle perimeter of twenty. Thus, there is some evidence of Function 5. As with Group One, if we look ahead to Segment b, we may say that the explanation in the first part led the students to explore what might be all possible solutions. Thus, Function 4 is present as well. Regarding rationality, once again we see students working towards a goal of finding two numbers whose sum is ten (TR). The procedure is correct (ER) and, as opposed to the first two groups, it is communicated with necessary details, such as stating that when the sum of the two sides is multiplied by 2, the result is 20 (CR). In addition, the communication employs relevant mathematical language using terms such as "sum" and "perimeter".

Segment *b*: As with the first group, both Function 1 and Function 2 of explanations are present. They describe what they did but they also explain why they do not include the square in their results, writing, "it is not a rectangle, but a square." From an ER point of view, the students' claim is incorrect. From a TR point of view, the students are working towards a goal, that is, they wish to show why only some examples are possible. They work towards this goal, ultimately reaching the example of the square, which seems to be, for them, the limit of the possibilities. Their written mathematical expressions, such as "6cm+4cm=x2" do not conform to acceptable mathematical convention (CR).

Group Four

This group of three students wrote the following:

We found different ways of [drawing] rectangles. For instance 9cm1cm x2 times so we got 20 cm. Other examples are:

Bartek: 2 cm, 8 cm; 6 cm, 4 cm; 3 cm, 7 cm

Angelo: 6cm, 3.5 cm; 8 cm, 2 cm; 10.5 cm, 1.2 cm; 7 cm, 3 cm

Manuel: 4 cm, 6 cm; 7 cm, 3 cm; 9 cm, 1 cm; 8 cm, 2 cm

As opposed to the other groups presented above, these students merely stated that they found different rectangles, some of which were incorrect, without describing the

procedure or explaining the method that they used to find them. Thus, Function 1 is not present and ER is lacking. Their means of communication is deficient "9cm1cm x2 times" (CR). Finally, their list of examples seems to lack direction and purpose. Thus TR is missing as well. In comparison to the other groups, Group Four's results are lacking both in terms of the functions of explanations and rationality. Perhaps the lack of functionality of their explanation is related to their lack of rationality.

DISCUSSION

This paper explored the possibility of extending the framework of rationality to the process of explaining. We found that the three components of rationality may be found in students' explanations and that the three components may be expressed or described differently for explanations with different functions. We propose a first refinement of the framework of rationality, with reference to some functions of explanation.

Function 1: *When explaining one's procedure, ER relates to the correctness of the procedure (from a mathematical point of view), CR to the communication of the method (all passages must be communicated), TR to mentioning the final goal of the procedure and possibly to the links between aims and actions*.

Function 2: *When explaining why a statement holds true, ER refers to the mentioning of correct mathematical properties, TR to the choice of suitable properties according to the proving aim, and CR to the organization of an intelligible explanation*.

Function 4: *When looking for all the possible solutions, and explaining why they are finite/infinite, ER relates to referencing mathematical properties, TR to mentioning the final goal and to the organization of the exploration and explanation accordingly (for instance, showing all the possible sums in a regular order). CR refers both to the communication of the exploration and of the final explanation*.

Function 5: *When justifying the plausibility of a method/procedure, ER refers to referencing correct mathematical properties and CR to the communication of the explanation in an intelligible form. At this point, perhaps due to the nature of the task, we have not been able to describe TR as it relates to Function 5. Additional research is needed regarding this aspect of Function 5*.

What emerges from the above analysis is that the three dimensions of rationality are always present, and that each function of explanation requires rationality in action. Furthermore, we found that a single task may elicit explanations with different functions and that the shift from one explanation to another may be "natural", occurring even when explicit requirements to give an explanation are missing. As was shown, some explanations (Function 1) may pave the way to further explanations (Function 2, 4 and 5). These findings suggest that explanations with different functions may help promote students' rationality.

The interest in combining frameworks is twofold: reaching a better understanding of the teaching episode at issue and improving theoretical frameworks (Prediger et al., 2008). Combining the frameworks, allowed us to fine tune our analysis in terms of

understanding how students' explanations may be intertwined with rationality. For example, as was shown in Groups 1 and 3, Function 5 of explanation seems to occur when there is a good level of TR evident in Function 1 of explanation. For the moment, we confined our analysis to the written group productions. The next step will be to use the refined and combined framework to analyse other parts of the teaching experiment, such as classroom discussions. From the point of view of functions of explanation, we expect that some functions of explanation, such as Function 1, 5 and 6 will emerge strongly in classroom discussions, while from the point of view of rationality, CR should have a major role, since communication to others is essential in discussions. An additional development concerns the planning and implementation of new tasks explicitly aimed at promoting different functions of explanation and associated rationality.

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